

# Super-radiance, Berry phase, Photon phase diffusion and Number squeezed state in the $U(1)$ Dicke ( Tavis-Cummings ) model

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Recently, strong coupling regimes of superconducting qubits or quantum dots inside a micro-wave circuit cavity and BEC atoms inside an optical cavity were achieved experimentally. The strong coupling regimes in these systems were described by the Dicke model. Here, we solve the Dicke model by a  $1/N$  expansion. In the normal state, we find a  $\sqrt{N}$  behavior of the collective Rabi splitting. In the superradiant phase, we identify an important Berry phase term which has dramatic effects on both the ground state and the excitation spectra of the strongly interacting system. The single photon excitation spectrum has a low energy quantum phase diffusion mode in *imaginary time* with a large spectral weight and also a high energy optical mode with a low spectral weight. The photons are in a number squeezed state which may have wide applications in high sensitive measurements and quantum information processing. Comparisons with exact diagonalization studies are made. Possible experimental schemes to realize the superradiant phase are briefly discussed.

Recently, several experiments [1] successfully achieved the strong coupling of a BEC of  $N \sim 10^5$   $^{87}\text{Rb}$  atoms to the photons inside an ultrahigh-finesse optical cavity. In parallel, strong coupling regime was also achieved with artificial atoms such as superconducting qubits inside micro-wave circuit cavity and quantum dots inside a semi-conductor micro-cavity system [3]. In these experiments, the individual maximum coupling strength  $\tilde{g}$  between the ( artificial ) atoms and field is larger than the spontaneous decay rate of the upper state  $\gamma$  and the intra-cavity field decay rate  $\kappa$ . The collective Rabi splitting was found to scale as  $\sqrt{N}$ . All these systems are described by the Dicke model[4] Eqn.1 where a single mode of photons coupled to an assembly of  $N$  atoms with the same coupling strength  $\tilde{g}$ .

The importance of various kinds of Dicke models in quantum optics ranks the same as the boson Hubbard model, Fermionic Hubbard model, Heisenberg model in strongly correlated systems and the Ising model in Statistical mechanics. Since the Dicke model was proposed in 1954, it was solved in the thermodynamic limit  $N = \infty$  by various methods [5–9]. It was found that when the collective atom-photon coupling strength is sufficiently large ( Fig.1 ), the system gets into a new phase called super-radiant phase where there are large number of inverted atoms and also large number photons in the system's ground state [11]. However, so far, there are only a few very preliminary exact diagonalization (ED) study on Dicke models at finite  $N$  [7, 9], its underlying physics remains unexplored [10]. It is known that any real symmetry breaking happens only at the thermodynamic limit  $N \rightarrow \infty$ , so in principle, there is no real symmetry breaking, so no real super-radiant phase at any finite  $N$ . But there is a very important new physics for a finite system  $N$  called quantum phase diffusion in *imaginary time* at

finite  $N$  for a continuous symmetry breaking ground state at  $N = \infty$ . The quantum phase diffusion process in a finite system is as fundamental and universal as symmetry breaking in an infinite system. Here, we will explore the quantum phase diffusion process of the Dicke model by a  $1/N$  expansion. We determine the ground state and single photon excitation spectrum in both normal and superradiant phase. In the normal state, we find a  $\sqrt{N}$  behavior of the collective Rabi splitting in the single photon excitation spectrum consistent with the experimental data and also determine the corresponding spectral weights. In the superradiant phase, we identify a Berry phase term which has dramatic effects on both ground state and the excitation spectra. The single photon excitation spectrum has a very low energy quantum phase diffusion mode  $E_D$  with a high spectral weight and also a high energy optical mode  $E_o$  with a low spectral weight. Their energies and the corresponding spectral weights are calculated. The photons are in a number squeezed state. The squeezing parameter ( namely, the Mandel  $Q_M$  factor ) is determined. It is the Berry phase which leads to the "Sidney Opera " shape in the single photon excitation spectrum and the consecutive plateaus in photon numbers in Fig.1. The Berry phase is also vital to make quantitative comparisons between the analytical results in this paper and the very preliminary ED results in [7] and much more extensive ED in [16]. Being very strong in intensity and has much enhanced signal/noise ratio, the number squeezed state from the superradiant phase may have wide applications in quantum information processing [13] and also in the field of high resolution and high sensitive measurement [14]. Several experimental schemes to realize the superradiant phase of the  $U(1)$  Dicke model briefly discussed.

In the  $U(1)$  Dicke model [4], a single mode of photons

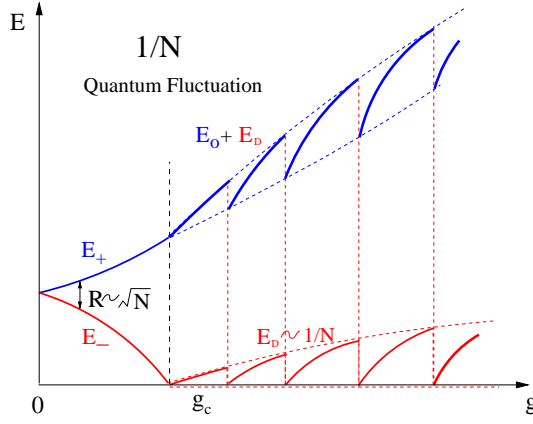


FIG. 1: ( Color online ) The single particle excitation spectrum  $E$  ( in energy unit ) of the  $U(1)$  Dicke ( Tavis-Cummings ) model Eqn.1 versus the collective atom-photon coupling strength  $g$  ( in energy unit ). The critical coupling is  $g_c = \sqrt{\omega_c \omega_a}$ . In the normal state  $g < g_c$ , the single photon spectrum  $S(\omega)$  has two peaks at the lower branch  $E_-$  and the upper branch  $E_+$  with spectral weights  $c_- \sim 1$  and  $c_+ \sim 1$  respectively. The collective Rabi splitting  $R = E_+ - E_- \sim \tilde{g}\sqrt{N}$  was measured in [1]. In the super-radiant phase  $g > g_c$ , the zero mode indicated by the dashed red line at  $g > g_c$  is lifted to the phase diffusion mode  $E_D$  in Eqn.10 by the  $1/N$  correction. The single photon spectrum  $S(\omega)$  has a peak at the low frequency phase diffusion mode  $E_D \sim \omega_c/N(\frac{1}{2} + \alpha)$  with the spectral weight  $c_D \sim N\lambda^2 \sim N$  and a high frequency optical mode  $E_o^t = E_o + E_D$  with a spectral weight  $c_o \sim 1$ . It is the Berry phase (  $-1/2 < \alpha < 1/2$  ) effects which lead to the " Sidney Opera " shape of the single photon energy spectrum.

couple to  $N$  two level atoms with the same coupling constant  $\tilde{g}$ . The two level atoms can be expressed in terms of 3 Pauli matrices  $\sigma_\alpha, \alpha = 1, 2, 3$ . Under the Rotating Wave (RW) approximation, the  $U(1)$  Dicke model can be written as:

$$H_{U(1)} = \omega_c a^\dagger a + \frac{\omega_a}{2} \sum_{i=1}^N \sigma_i^z + \frac{g}{\sqrt{N}} \sum_{i=1}^N (a^\dagger \sigma_i^- + h.c.) \quad (1)$$

where the  $\omega_c, \omega_a$  are the cavity photon frequency, the energy difference of the two atomic levels respectively, the  $g = \sqrt{N}\tilde{g}$  is the collective photon-atom coupling (  $\tilde{g}$  is the individual photon-atom coupling ), the cavity mode  $a$  could be any one of the two orthogonal polarizations of  $TEM_{00}$  cavity modes in [1]. One can also add the atom-atom interaction  $H_{at-at}$  to the Eqn.1. Because  $H_{at-at}$  does not change the symmetry of the model, we expect all the results achieved in the paper remain qualitatively valid. The Hamiltonian Eqn.1 has the  $U(1)$  symmetry  $a \rightarrow ae^{i\theta}, \sigma^- \rightarrow \sigma^- e^{i\theta}$ . In the normal phase,  $\langle a \rangle = 0$ , the  $U(1)$  symmetry is respected. In the super-radiant phase,  $\langle a \rangle \neq 0$ , the  $U(1)$  symmetry is spontaneously broken. The model Eqn.1 was studied by a large  $N$  expansion in Ref.[6, 8]. However, they did not extract any important physics at the order of  $1/N$ . In this paper,

we will show that by carefully analyzing the effects of the zero mode in the super-radiant phase, one can extract the most important physics of quantum phase diffusion at the order  $1/N$ . In the large  $N$  expansion in the magnetic systems [15],  $N$  is the order of the magnetic symmetry group  $O(N), SU(N), Sp(2N)$  with  $N = 3, 2, 1$  respectively. However here  $N \sim 10^5$  is the number of atoms, so  $1/N \sim 10^{-5}$  expansion is quite accurate.

Following standard large  $N$  techniques developed in [15], after re-scaling the photon field  $a \rightarrow \sqrt{N}a$ , integrating out the spin degree of freedoms, one can get an effective action  $S_{eff}[a]$  in terms of the photon field only, then perform a large  $N$  expansion. At  $N \rightarrow \infty$ , the photon mean field value  $\langle a(\tau) \rangle = \lambda$  is determined [4-7] by the saddle point equation:  $\omega_c \lambda = g^2 \lambda \frac{\tanh \beta E}{2E}$  where  $E = \sqrt{(\frac{\omega_a}{2})^2 + g^2 \lambda^2}$  and  $\beta = 1/k_B T$  is the inverse temperature. At  $T = 0$ , in the normal phase  $g < g_c = \sqrt{\omega_a \omega_c}$ ,  $\lambda = 0$ , the photon number  $n/N = 0$ ; in the super-radiant phase  $g > g_c$ ,  $\lambda \neq 0$ , the photon number  $n/N = \lambda^2 \sim g - g_c$  when  $g$  is slightly above  $g_c$ . Its phase diagram and photon number at  $N = \infty$  and finite  $N$  is shown in Fig.1 and 2 respectively.

At finite  $N$ , writing  $a = \lambda + \psi$  where  $\psi$  describes the photon fluctuation around its mean field value  $\lambda$ , one can expand the  $S[a] = S_0 + S_2 + S_3 + \dots$  to second order [8] in  $\psi$ :

$$S_2[\bar{\psi}, \psi] = \frac{N}{2\beta} \sum_{i\omega} (\bar{\psi}(\omega), \psi(-\omega)) G^{-1} \begin{pmatrix} \bar{\psi}(\omega) \\ \psi(-\omega) \end{pmatrix} \quad (2)$$

$$G^{-1} = \begin{pmatrix} K_1 & K_2 \\ K_2^* & K_1^* \end{pmatrix}$$

where the explicit expressions of the  $K_1, K_2$  are given in [8], but are not needed in the following.

In the normal phase  $g < g_c$ ,  $\lambda = 0$ . One can see the normal Green function at  $T = 0$  in Eqn.2 has two poles  $E_{\pm} = \frac{(\omega_c + \omega_a) \pm \sqrt{(\omega_c - \omega_a)^2 + 4g^2}}{2}$  with the spectral weights  $c_+ = \frac{E_+ - \omega_a}{E_+ - E_-}, c_- = \frac{\omega_a - E_-}{E_+ - E_-}$  ( Fig.1 ). After the analytic continuation  $\tau \rightarrow it$ , the one photon Green function at  $T = 0$  takes:

$$\langle a(t) a^\dagger(0) \rangle_N \sim c_- e^{-iE_- t} + c_+ e^{-iE_+ t} \quad (3)$$

where we also put back the re-scaling factor of the photon field  $a \rightarrow \sqrt{N}a$ . It leads to the two peaks at the two poles  $E_{\pm}$  in the single photon energy spectrum shown in the Fig.1. At the resonance  $\Delta_c = \omega_c - \omega_a = 0$ , the collective Rabi splitting  $E_+ - E_- = 4g = 4\sqrt{N}\tilde{g} \sim \sqrt{N}$  shown in Fig.1 was measured in [1]. Note that due to  $\tilde{g}_{\sigma_+} > \tilde{g}_{\sigma_-}$  in [1], the coefficients of the  $\sqrt{N}$  are different for the two different polarizations. The intensity ratio of the two peaks  $\frac{c_+}{c_-} = \frac{E_+ - \omega_a}{\omega_a - E_-}$  seems has not been measured yet in [1].

However, at the super-radiant phase  $g > g_c$ ,  $\lambda \neq 0$ . the anomalous term  $K_2 \neq 0$  and  $|K_1|^2 - |K_2|^2$  contains a zero

mode shown as a red dashed line in Fig.1, in addition to the pole at a high frequency  $E_o = \sqrt{(\omega_c + \omega_a)^2 + 4g^2\lambda^2}$  shown as the blue dashed line in Fig.1. This "zero" mode is nothing but the "Goldstone" mode due to the global  $U(1)$  symmetry breaking in the super-radiant phase. The important physics behind this "zero" mode was never addressed in the previous literatures [6, 8]. Here we will explore the remarkable properties of this "zero" mode. Because of the infra-red divergences from this zero mode, the  $1/N$  expansion in the Cartesian coordinates need to be summed to infinite orders to lead to a finite physical result. It turns out that the non-perturbative effects of the zero mode can be more easily analyzed in the polar coordinate ( or phase representation ) by writing  $a = \lambda + \psi_1 + i\psi_2 = \sqrt{\lambda^2 + \delta\rho}e^{i\theta}$ , then to linear order in  $\delta\rho$  and  $\theta$ :  $\psi_2 = \lambda\theta$ . In Eqn.2, by integrating out the massive  $\psi_1$  mode, using  $\psi_2 = \lambda\theta$ , also paying a special attention to the Berry phase term [17] coming from the angle variable  $\theta$ , one can show that the dynamics of the phase  $\theta$  is given by:

$$S_2[\theta] = iN\lambda^2\partial_\tau\theta + \frac{N}{2\beta} \sum_{i\omega} \frac{2\lambda^2\omega^2(\omega^2 + E_o^2)}{\omega_c(\omega^2 + 4g^2\lambda^2)} |\theta(\omega)|^2 \quad (4)$$

In the following, we will discuss the the zero mode and the optical mode respectively.

In the low frequency  $\omega \ll E_o$  limit where the magnitude fluctuations can be dropped, Eqn.4 reduces to:

$$\mathcal{L}_{PD}[\theta] = iN\lambda^2\partial_\tau\theta + \frac{1}{2D}(\partial_\tau\theta)^2 = \frac{1}{2D}(\partial_\tau\theta + i\alpha D)^2 \quad (5)$$

with the quantum phase diffusion constant  $D = \frac{2\omega_c g^2}{E_o^2 N}$ .

In the Eqn.5, we have denoted  $N\lambda^2 = N_0 + \alpha$  where  $N_0 = [N\lambda^2]$  is the *closest* integer to  $N\lambda^2$ , so  $-1/2 < \alpha < 1/2$ .

The corresponding quantum phase diffusion Hamiltonian is:

$$H_{PD}[\theta] = \frac{D}{2}(\delta N_{ph} - \alpha)^2 \quad (6)$$

where  $\delta N_{ph} = N_{ph} - N_0$  is the photon number fluctuation around its ground state value  $N_0$  and is conjugate to the phase  $\theta$ :  $[\theta, \delta N_{ph}] = i\hbar$ . In fact, Eqn.6 can be considered as the Hamiltonian of a particle moving along a ring with a very large inertial of moment  $I = 1/D$  subject to a fractional flux  $f = \phi/\phi_0 = \alpha$ .

In Eqn.5, after defining  $\tilde{\theta}(\tau) = \theta(\tau) + i\alpha D\tau$ , one can easily show that

$$\langle (\tilde{\theta}(\tau) - \tilde{\theta}(0))^2 \rangle = 2D \int \frac{d\omega}{2\pi} \frac{1 - e^{i\omega\tau}}{\omega^2} = D|\tau| \quad (7)$$

which is a phase diffusion in *imaginary time*  $\tau$  [12, 18] with the phase diffusion constant  $D$ . Only in the thermodynamic limit  $N \rightarrow \infty$ , a state with a given initial phase will stick to this phase as the time evolves, so we have a spontaneously broken  $U(1)$  symmetry. However, for any

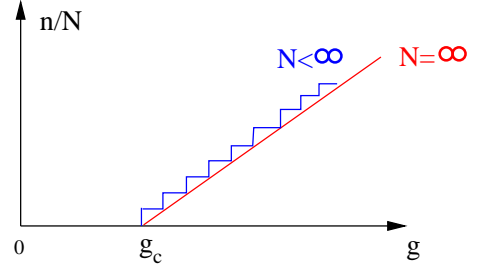


FIG. 2: ( Color online ) The average photon number in the ground state at  $N = \infty$  and at a finite  $N < \infty$  versus the collective atom-photon coupling strength  $g$  ( in energy unit ). The photon is very close to be the Fock state as shown by the steps in the superradiant phase at  $N < \infty$ . It is the Berry phase (  $-1/2 < \alpha < 1/2$  ) effects which lead to the steps in the photon number.

finite  $N$ , the initial phase has to diffuse with the phase diffusion constant  $D \sim 1/N$ . The diffusion time scale in the *imaginary time* beyond which there is no more phase coherence is  $\tau_D = 1/D \sim N/\omega_c$  which is finite for any finite  $N$ . This can also be called phase "de-coherence" time in the *imaginary time* [18].

From Eqn.5, it is easy to see that the gapless nature of the phase diffusion mode in Eqn.5 leads to the vanishing of the order parameter inside the super-radiant phase:

$$\langle a \rangle = 0 \quad (8)$$

So the  $U(1)$  symmetry is restored by the phase diffusion. From Eqn.5, after doing the analytic continuation  $\tau \rightarrow it$ , we can get:

$$\langle a^\dagger(t)a(0) \rangle_S = N\lambda^2 e^{-i(\frac{1}{2} + \alpha)Dt} \quad (9)$$

where we also put back the re-scaling factor of the photon field  $a \rightarrow \sqrt{N}a$ . It is also easy to see that  $\langle a(t)a(0) \rangle_S = 0$ , so there is no quadrature squeezing anymore at any finite  $N$  [19]. In fact, all these results can also be achieved by using the Hamiltonian Eqn.6.

Eqn.9 leads to the result that the energy of the "zero energy mode" ( Goldstone mode ) at  $N = \infty$  was "lifted" to a quantum phase "diffusion" mode at any finite  $N$  with a finite small positive frequency [18]:

$$E_D = (\frac{1}{2} + \alpha)D = \frac{2\omega_c g^2 (\frac{1}{2} + \alpha)}{[(\omega_c + \omega_a)^2 + 4g^2\lambda^2]N} \sim \omega_c/N \quad (10)$$

It is the Berry phase effect which leads to the periodic jumps in the Fig.1. The Fourier transform of Eqn.9 leads to the Fluorescence spectrum  $S(\omega) = N\lambda^2 \delta(\omega - E_D)$  with the spectral weight  $c_D^0 = N\lambda^2 \sim N$ .

Now we study the photon statistics. If neglecting the magnitude fluctuation, the quantum phase diffusion Hamiltonian Eqn.6 shows that the ground state is a photon Fock state with eigenvalue  $N_0$  which jumps by 1 in all the plateaus ending at  $\alpha = 1/2$  in the Fig.2. Now

we incorporate the magnitude fluctuation. In the Eqn.2, by integrating out the imaginary part  $\psi_2(\omega)$  and using  $\psi_1 = \delta\rho/2\lambda$ , one can get the effective action for the magnitude fluctuations

$$\mathcal{L}_2(\delta\rho) = N \frac{\omega^2 + E_o^2}{8\lambda^2\omega_c} |\delta\rho(\omega)|^2 \quad (11)$$

where we find the Mandel factor  $Q_M = \frac{\langle(\delta N)^2\rangle - \langle N\rangle}{\langle N\rangle} = -1 + \frac{2\omega_c}{E_o}$  so the deviation from the Fock state at any given plateau in Fig.2 is given by  $\frac{2\omega_c}{E_o}$ . Because  $\frac{2\omega_c}{E_o} \ll 1$  in the  $g \gg g_c$  limit, it is very close to be a Fock state. This is a highly non-classical state with Sub-Poissonian photon statistics. It has very strong signal  $\langle N_{ph} \rangle = N\lambda^2$ , but nearly no photon number noise, so it has a very large signal to noise ratio which could be crucial for quantum information processing [13] and also in the field of high resolution and high sensitive measurement [14].

Using the polar representation  $a = \sqrt{\lambda^2 + \delta\rho} e^{i\theta} \sim \lambda e^{i\theta} + \frac{\delta\rho}{2\lambda} e^{i\theta} + O(1/N)$ , one can evaluate the photon correlation function:

$$\begin{aligned} \langle \mathcal{T} a^\dagger(\tau) a(0) \rangle &= N\lambda^2 \langle e^{-i(\theta(\tau) - \theta(0))} \rangle \\ &+ \frac{N}{4\lambda^2} \langle \delta\rho(\tau) \delta\rho(0) \rangle \langle e^{-i(\theta(\tau) - \theta(0))} \rangle + O(1/N) \end{aligned} \quad (12)$$

where the  $\mathcal{T}$  means imaginary time ordered. By evaluating the first and second term from Eqns.4 and 11, we can identify not only the quantum phase diffusion mode  $E_D = D(\frac{1}{2} + \alpha)$  with the corresponding spectral weight  $c_D = N\lambda^2 - \frac{\omega_c(\omega_c + \omega_a)^2}{4E_o^3}$ , but also the optical mode  $E_o^t = E_o + E_D$  with the corresponding spectral weight  $c_o = \frac{\omega_c(\omega_c + \omega_a)^2}{4E_o^3} + \frac{2\omega_c}{E_o}$  ( Fig.1 ). Note that  $E_o^t - E_D = E_o$  in Fig.1 is independent of the Berry phase  $\alpha$ . So the total energy in the optical frequency peak  $\sim E_o^t \times c_o \sim \omega_c$  is comparable to that in the phase diffusion mode  $\sim E_D \times c_D \sim \omega_c/N \times N \sim \omega_c$ .

If one introduce the total "spin" of the  $N$  two level atoms  $J^z = \sum_i \sigma_i^z$ ,  $J^+ = \sum_i \sigma_i^+$ ,  $J^- = \sum_i \sigma_i^-$  and confine the Hilbert space only to  $J = N/2$ , then Eqn.1 can be simplified to the  $J-U(1)$  Dicke model which was studied by an exact diagonalization (ED) in [7]. The authors in [7] found that there are a series of ground state energy level crossings as the  $g$  gets into the super-radiant regime and interpreted them as consecutive "quantum phase transitions". But they did not study any excited states. In [16], we performed a much more extensive ED study directly on the  $U(1)$  Dicke model Eqn.1 and not only calculated the ground states, but also all the excited energy levels. We also identified a series of ground state energy level crossings in the super-radiant regime. By comparing with the analytic results achieved in this paper, we found all these ground states crossings are precisely due to the periodic changes of the Berry phase  $\alpha$  in Eqns.4,5,6. They are not consecutive "quantum phase-like transitions" as claimed in [7]. We also found one to

one quantitative matches between the low energy phase diffusion mode  $E_D$ , also the high energy optical mode  $E_o^t = E_o + E_D$  and the excited levels found by the ED in [16] at  $N$  as small  $N = 5$ . The complete comparisons will be presented in [16].

It remains experimentally challenging to move into the superradiant regime which requires the collective photon-atom coupling  $g = \sqrt{N}\tilde{g} > g_c = \sqrt{\omega_c\omega_a} \sim 2\pi \times 10^5 GHz$  for the optical cavity used in [1]. The collective Rabi splitting  $\sqrt{N}\tilde{g} \sim 20GHz$  in [1] is still much smaller than  $\omega_c = 2\pi \times 10^5 GHz$ , so not even close to the superradiant regime in Fig.1. It was proposed in [20–22] that the super-radiant regime can be realized by using a cavity-plus-laser-mediated Raman transitions between a pair of stable atomic ground states, therefore also suppress the spontaneous emission  $\gamma$ . All the parameters in Eqn.1 can be controlled by the external laser frequencies and intensities, so the characteristics energy scales in the effective two level atoms are no longer those of optical photons and dipole coupling, but those associated with Raman transition rates and light shifts. Indeed, using this scheme, the super-radiant phase in the  $Z_2$  Dicke model [9, 20] was reached by using both thermal atoms [21] and the cold atoms in the BEC [22]. We expect this scheme may also be used to realize the super-radiant phase of the  $U(1)$  Dicke model shown in Fig.1. Because the microwave circuit cavity has much lower cavity frequency and the individual photon-qubit  $\tilde{g}$  can also be made very large, so the superradiant phase could also be realized in superconducting qubits or quantum dots inside a circuit cavity in the future. In the experiments, there is also a weak dissipation  $\kappa \ll g$ . In a future publication, following the procedures in [19], we will study the effects of  $\kappa$  on the number squeezed state.

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- [1] F. Brennecke, *et. al* Nature 450, 268 ( 2007), Yves Colombe, *et al* Nature 450, 272 ( 2007).
  - [2] In fact, the  $\sqrt{N}$  behaviour was observed in thermal atoms before, M.G.Raizen, *et al*, PRL, 63, 240(1989); S. Leslie, *et. al*, Phys. Rev. A, 69, 043805 (2004); Xudong Yu, *et al*, Phys. Rev. A 79, 061803(R) (2009). However, the superradiant phase can be more easily realized in the cold

- atom BEC systems as demonstrated in the very recent experiment in [22].
- [3] A. Wallraff, *et al*, Nature 431, 162 (2004). J. M. Fink, *et al*, PRL 103, 083601 (2009); Lev S. Bishop, *et al*, Nature Physics 5, 105-109 (2009); J. P. Reithmaier, *et al*, Nature 432, 197 (2004). T. Yoshie, *et al*, Nature 432, 200 (2004). G. Gunter, *et al*, Nature, 458, 178 (2009).
  - [4] R. H. Dicke, Phys. Rev. 93, 99 (1954).
  - [5] K. Hepp and E. H. Lieb, Anns. Phys. ( N. Y. ), 76, 360 (1973); Y. K. Wang and F. T. Hioe, Phys. Rev. A, 7, 831 (1973).
  - [6] V. N. Popov and S. A. Fedotov, Soviet Physics JETP, 67, 535 (1988); V. N. Popov and V. S. Yarunin, Collective Effects in Quantum Statistics of Radiation and Matter (Kluwer Academic, Dordrecht, 1988).
  - [7] V. Buzek, M. Orszag and M. Roko, Phys. Rev. Lett. 94, 163601 (2005).
  - [8] P. R. Eastham and P. B. Littlewood, Phys. Rev. B 64, 235101 (2001).
  - [9] For a different kind of Dicke model with a  $Z_2$  symmetry, see C. Emary and T. Brandes, Phys. Rev. Lett. 90, 044101 (2003); Phys. Rev. E 67, 066203 (2003). N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. 92, 073602 (2004). For its possible recent experimental realization, see [21, 22]. Due to the very different symmetries, the  $Z_2$  Dicke model at a finite  $N$  has completely different properties than the  $U(1)$  model at a finite  $N$  and will be presented in a separate publication.
  - [10] The  $U(1)$  Dicke ( Tavis-Cummings ) model is integrable at any finite  $N$ , so, in the " face " value, the system's eigen-energy spectra could be "exactly" solvable by Bethe Ansatz like methods. For example, see N.M. Bogoliubov, R.K. Bullough, and J. Timonen, Exact solution of generalized Tavis-Cummings models in quantum optics, J. Phys. A: Math. Gen. 29 6305 (1996). However, so far, the Bethe Ansatz like solutions stay at very "formal" level from which it is even not able to get the system's eigen-energy analytically, let alone to extract any underlying physics. Furthermore, it is well known the Bethe Ansatz method is not able to get any dynamic correlation functions.
  - [11] In this paper, we study the *steady state* super-radiant phase which involves quantum phase transition. It is quite different from the conventional *non-equilibrium* spontaneous super-radiant radiation which can be understood from a simple Golden rule calculation, see Chap.6.7 in Y. Yamamoto and A. Imamoglu, Mesoscopic quantum optics, John Wiley & Sons, Inc. 1999. For the recent advances on the superradiance prepared by a single photon absorption, see A. A. Svidzinsky, J.T. Chang, M. O. Scully, Phys. Rev. Lett. 100, 160504 (2008); M. O. Scully, Phys. Rev. Lett. 102, 143601 (2009), M. O. Scully and A. A. Svidzinsky, Science, 325, 1510 (2009).
  - [12] The well studied "phase diffusion" process of cold atoms BEC always involves a sudden quenching of the Hamiltonians of the system, it leads to the collapse and revival in the real time. However, the width of the phase distribution increases linearly in the real time  $t$ , instead of  $\sqrt{t}$ , so the phrase " phase diffusion" may not be a suitable one in this context. See Eqn.7 and [18] for comparisons. See E. M. Wright and D. F. Walls, J. C. Garrison, Phys. Rev. Lett. 77, 2158, (1996); M. Lewenstein and L. You, *ibid*, 77, 3489 (1996); A. Imamoglu, M. Lewenstein and L. You, *ibid*, 78, 2511 (1997). J. Javanainen and M. Wilkens, *ibid*, 78, 4675 (1997); M. Greiner, O. Mandel, T.W.Hansch and I. Bloch, Nature 419, 51 (2002).
  - [13] L.Davidovich, Rev. Mod. Phys. 68, 127(1996).
  - [14] C. F. Wildfeuer, *et.al*, Phys. Rev. A 80, 043822 (2009).
  - [15] J. Ye and S. Sachdev, Phys. Rev. B 44, 10173 (1991). A. Chubukov, S. Sachdev and Jinwu Ye, Phys.Rev.B, 11919 (1994).
  - [16] Jinwu Ye, *et.al*, in preparation.
  - [17] We are indebted to B. Halperin and S. Sachdev for stressing the importance of this Berry phase term.
  - [18] Note that the classical phase diffusion due to random spontaneous emission events in a Laser is in *real time* and leads to natural linewidth of a Laser, see Eqn.11.4.8 in M. O. Scully and M. S. Zubairy, Quantum Optics, Cambridge University press, 1997. Because the Berry phase is a completely quantum effect, so there is no Berry phase analog in this classical phase diffusion process.
  - [19] For a two mode quadrature squeezing of photons emitted from an exciton superfluid, see Jinwu Ye, T. Shi and Longhua Jiang, Phys. Rev. Lett. 103, 177401 (2009); T. Shi, Longhua Jiang and Jinwu Ye, Phys. Rev. B 81, 235402 (2010).
  - [20] F. Dimer, *et.al*, Phys. Rev. A 75, 013804 (2007).
  - [21] A. T. Black, H. W. Chan and V. Vuletic, Phys. Rev. Lett. 91, 203001(2003).
  - [22] K. Baumann, *et.al*, Nature 464, 1301-1306 (2010).